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Hamilton–Jacobi Expansion of the Scattering Amplitude

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The calculation of the scattering amplitude is reduced to the problem of solving a set of classical Hamilton–Jacobi equations. This allows one to incorporate classical intuition into approximations at a fundamental level. The result is actually an iterative expansion for the scattering amplitude which is expected to be convergent in the high-energy limit. The first term in the expansion is shown to be the Glauber approximation, which is an approximation used extensively in nuclear as well as atomic and particle physics.

1. INTRODUCTION

The exact solution to Schrödinger's equation and, hence, to the scattering amplitude for an arbitrary potential has not been found. One, therefore, inevitably has to resort to some type of an approximation scheme for most scattering problems. Without doubt, approximations can be more easily made and understood in terms of classical mechanical arguments. In this work the calculation of the scattering amplitude is reduced to an iterative expansion which requires only that a set of classical Hamilton–Jacobi equations be solved. This can allow one to incorporate classical approximations in a very fundamental way into the solution of the quantum scattering amplitude.

The outline of this work is as follows. In the next section it will be shown how the calculation of the scattering amplitude is related to Schrödinger's equation. In Section 3, the Hamilton–Jacobi iterative solution will be formulated. It is shown that this is an expansion in terms of Planck's constant and is expected to be most useful for high-energy scattering. Finally, in Section 4, it is shown that the first term in this expansion can easily lead to the well-known Glauber form of the eikonal approximation

(Glauber, 1959). Successive terms, therefore, can be viewed as corrections to the Glauber approximation.

2. THE SCATTERING AMPLITUDE AND SCHRÖDINGER'S EQUATION

Consider the scattering of a single, structureless particle from a local potential V. The Hamiltonian for the system is

$$H = H_0 + V \tag{1}$$

Where H_0 is the free particle Hamiltonian. If the incident particle has momentum $\mathbf{p}_0$ (energy E_0), and the scattered particle has momentum $\mathbf{p}$ (energy E), the relevant S-matrix element is (Taylor, 1972)

$$\langle \mathbf{p}|S|\mathbf{p}_0\rangle = e^{(i/\hbar)Et}e^{(-i/\hbar)E_0 t_0}\int \frac{d\mathbf{x}\,d\mathbf{x}_0}{|2\pi i\hbar|^2} \times e^{(-i/\hbar)\mathbf{p}\cdot\mathbf{x}}\langle \mathbf{x}|e^{(-i/\hbar)H(t-t_0)}|\mathbf{x}_0\rangle e^{(i/\hbar)\mathbf{p}_0\cdot\mathbf{x}_0} \tag{2}$$

In this equation and throughout this paper, the limits $t\to\infty$ and $t_0\to-\infty$ are always to be understood as they refer to the final and initial times of the scattering event, respectively.

The matrix element on the right-hand-side of equation (2) is precisely the kernel for Schrödinger's equation (Merzbacher, 1961). That is,

$$K(\mathbf{x},t;\mathbf{x}_0,t_0) = \langle \mathbf{x}|e^{(-i/\hbar)H(t-t_0)}|\mathbf{x}_0\rangle \tag{3}$$

is the kernel, and it can be used to determine the wave function of the system at time t given the wave function at time t_0 via the relation

$$\psi(\mathbf{x},t) = \int d\mathbf{x}_0\, K(\mathbf{x},t;\mathbf{x}_0,t_0)\psi(\mathbf{x}_0,t_0) \tag{4}$$

The wave function $\psi(\mathbf{x},t)$, of course, must satisfy Schrödinger's differential equation:

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi(\mathbf{x},t) = i\hbar\frac{\partial\psi(\mathbf{x},t)}{\partial t} \tag{5}$$

By requiring that the wave function of equation (4) satisfy Schrödinger's equation, it is easily shown [since $\psi(\mathbf{x}_0,t_0)$ is arbitrary] that the kernel itself satisfies Schrödinger's equation in the variables $\mathbf{x}$ and t:

$$\left(\frac{-\hbar^2}{2m}\nabla^2+V\right)K(\mathbf{x},t;\mathbf{x}_0,t_0)=i\hbar\frac{\partial K(\mathbf{x},t;\mathbf{x}_0,t_0)}{\partial t} \tag{6}$$

Hence, the calculation of the S-matrix element is reduced to solving Schrödinger's equation for the kernel.

The scattering amplitude is related to the S-matrix as follows (Taylor, 1972). The scattering process is looked at from the wave packet point of view with $|\phi\rangle$ representing the "in" asymptotic wave packet. Its momentum representation $\langle \mathbf{p}'|\phi\rangle \equiv \phi_{\mathbf{p}_0}(\mathbf{p}')$ is assumed to be well peaked about a given initial momentum $\mathbf{p}_0$. That is, $\phi_{\mathbf{p}_0}(\mathbf{p}')$ is negligible except where $\mathbf{p}'\simeq \mathbf{p}_0$. Also needed is the state $|\phi_\rho\rangle$ which represents the in asymptotic state which has been rigidly displaced by an amount $\boldsymbol{\rho}$ in coordinate space. The plane of the vectors $\boldsymbol{\rho}$ is perpendicular to $\mathbf{p}_0$. $\langle \mathbf{p}'|\phi_\rho\rangle$ is then given by

$$\langle \mathbf{p}'|\phi_\rho\rangle = e^{-i\rho\cdot\mathbf{p}/\hbar}\langle \mathbf{p}'|\phi\rangle = e^{-i\mathbf{p}\cdot\rho/\hbar}\phi_{\mathbf{p}_0}(\mathbf{p}') \tag{7}$$

The probability $\omega(d\Omega\leftarrow|\phi_\rho\rangle)$ that the particle represented by the incoming state $|\phi_\rho\rangle$ scatters into the solid angle $d\Omega$ about the final momentum direction $\hat{p}$ is given in terms of the out asymptotic state $\langle \mathbf{p}|\psi_{\text{out}}\rangle \equiv \psi_{\text{out}}(\mathbf{p})$ by

$$\omega(d\Omega\leftarrow|\phi_\rho\rangle) = d\Omega\int_0^\infty p^2\,dp\,|\psi_{\text{out}}(\mathbf{p})|^2 \tag{8}$$

The S operator, by definition, relates $|\psi_{\text{out}}\rangle$ to $|\psi_{\text{in}}\rangle = |\phi_\rho\rangle$ via the relation

$$\langle \mathbf{p}|\psi_{\text{out}}\rangle = \int d\mathbf{p}'\langle \mathbf{p}|S-1|\mathbf{p}'\rangle\langle \mathbf{p}'|\phi_\rho\rangle \tag{9}$$

or using equation (7)

$$\psi_{\text{out}}(\mathbf{p}) = \int d\mathbf{p}'\langle \mathbf{p}|S-1|\mathbf{p}'\rangle e^{-ip'\cdot\rho/\hbar}\phi_{p_0}(\mathbf{p}') \tag{10}$$

The cross section for the scattering process is given by

$$\sigma(d\Omega\leftarrow|\phi\rangle) = \int d^2\rho\,\omega(d\Omega\leftarrow|\phi_\rho\rangle)$$

$$= d\Omega\int p^2\,dp\,d^2\rho\,|\psi_{\text{out}}(\mathbf{p})|^2 \tag{11}$$

Using equation (10), this becomes

$$\sigma = d\Omega \int p^2\,dp\,d^2\rho\,d\mathbf{p}'\,dp''$$

$$\times \langle \mathbf{p}|S-1|\mathbf{p}''\rangle^* \langle \mathbf{p}|S-1|\mathbf{p}'\rangle e^{i(\mathbf{p}''-\mathbf{p}')\cdot\rho} \phi^*_{\mathbf{p}_0}(\mathbf{p}'')\phi_{\mathbf{p}_0}(\mathbf{p}') \tag{12}$$

Given the S-matrix elements, one can carry out the above integration to obtain the cross section. The dependence upon the wave packets would eventually drop out of the expression after utilizing the normalization integral

$$\int d\mathbf{p}'\,\phi_{\mathbf{p}_0^*}(\mathbf{p}')\phi_{\mathbf{p}_0}(\mathbf{p}') = 1 \tag{13}$$

The scattering amplitude $f(\mathbf{p}, \mathbf{p}_0)$ can be extracted by inspection from the expression for the cross section via the definition

$$\sigma \equiv d\Omega |f(\mathbf{p}, \mathbf{p}_0)|^2 \tag{14}$$

This then relates the scattering amplitude to the S matrix and, hence, to Schrödinger's equation.

3. THE ITERATIVE METHOD—FORMULATION

In the last section, the calculation of the scattering amplitude was reduced to the problem of solving Schrödinger's equation for the kernel. The further reduction of this to a solution of only the classical Hamilton–Jacobi equation has its inspirational basis in the Feynman's path integral formulation of quantum mechanics (Feynman and Hibbs, 1965). In this formulation, the wave function of a system at time t can be obtained from that at time t_0 via the relation

$$\psi(\mathbf{x}, t) = \lim_{\epsilon \to 0} \int \exp\left[\frac{i}{\hbar} \sum_i S(\mathbf{x}_{i+1}, \mathbf{x}_i)\right] \psi(\mathbf{x}_0, t)\frac{d\mathbf{x}_0}{A}\frac{d\mathbf{x}_1}{A}\cdots\frac{d\mathbf{x}_{N-1}}{A} \tag{15}$$

Here, $\mathbf{x}_i$ is the position of the system at time t_i; $\mathbf{x}_{i+1}$ is the position at time $t_{i+1} = t_i + \varepsilon$; $\mathbf{x}_N = \mathbf{x}$; A is a normalization constant; and $S(\mathbf{x}_{i+1}, \mathbf{x}_i)$ is the classical action calculated from the time integral of the Lagrangian over the classical path from $\mathbf{x}_i$ to $\mathbf{x}_{i+1}$:

$$S(\mathbf{x}_{i+1},\mathbf{x}_i)=\int_{t_i}^{t_{i+1}} L(\dot{\mathbf{x}}(t),\mathbf{x}(t))\,dt \tag{16}$$

One can assume, in equation (19), that the integrals over $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_{N-1}$ have been done and the limit $\varepsilon\to 0$ taken. The result can, in general, be written in the form

$$\psi(\mathbf{x},t)=\int\frac{\exp[(i/\hbar)S(\mathbf{x},t;\mathbf{x}_0,t_0)]}{B(\mathbf{x},t;\mathbf{x}_0,t_0)}\psi(\mathbf{x}_0,t_0)\,d\mathbf{x}_0 \tag{17}$$

$S(\mathbf{x},t;\mathbf{x}_0,t_0)$ is the classical action for the entire path from $\mathbf{x}_0$ to $\mathbf{x}$, and it satisfies the Hamilton–Jacobi equation (Lanczos, 1966)

$$\frac{(\nabla S)^2}{2m}+V=\frac{-\partial S}{\partial t} \tag{18}$$

With V being the potential energy function of the problem.

Comparing equation (17) with equation (4), it is seen that the kernel is given by

$$K(\mathbf{x},t;\mathbf{x}_0,t_0)=\frac{\exp[(i/\hbar)S(\mathbf{x},t;\mathbf{x}_0,t_0)]}{B(\mathbf{x},t;\mathbf{x}_0,t_0)} \tag{19}$$

Requiring that this must satisfy Schrödinger's equation, one obtains the relation

$$\frac{(\nabla S)^2}{2m}+V-\frac{\hbar^2}{2m}\left(B\nabla^2B^{-1}+\frac{i}{\hbar}\nabla^2S+\frac{2i}{\hbar}B\nabla B^{-1}\cdot\nabla S\right)$$

$$=\frac{-\partial S}{\partial t}+i\hbar B\frac{\partial B^{-1}}{\partial t} \tag{20}$$

In view of equation (18), the equation which determines the function B is given by

$$\frac{-\hbar^2}{2m}B\nabla^2B^{-1}\frac{-i\hbar}{m}\left(B\nabla B^{-1}\cdot\nabla S+\frac{\nabla^2S}{2}\right)=i\hbar B\frac{\partial B^{-1}}{\partial t} \tag{21}$$

Hence, the calculation of the kernel and, therefore, the scattering amplitude has been reduced to solving the Hamilton–Jacobi equation of equation (18) for the action S and solving equation (21) for the function B. If the following definitions are made

$$B_1 \equiv B \tag{22}$$

$$S_0 \equiv S \tag{23}$$

and

$$V_1 \equiv \frac{-i\hbar}{m}\left(B_1 \nabla B_1^{-1} \cdot \nabla S_0 + \frac{\nabla^2 S_0}{2}\right) \tag{24}$$

then equation (15) becomes

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V_1\right) B_1^{-1} = i\hbar \frac{\partial B_1^{-1}}{\partial t} \tag{25}$$

Which is the Schrödinger equation for the function B_1. Hence, the calculation of the kernel which satisfies Schrödinger's equation has been written formally as the combination of the solution to the Hamilton–Jacobi equation and that of the Schrödinger equation again.

The latter equation [equation (25)] can, in turn, be written formally as the same combination of solutions. That is,

$$B_1^{-1} = e^{(i/\hbar)S_1}/B_2 \tag{26}$$

Where S_1 satisfies the Hamilton–Jacobi equation:

$$\frac{(\nabla S_1)^2}{2m} + V_1 = \frac{-\partial S_1}{\partial t} \tag{27}$$

and B_2 satisfies the Schrödinger equation

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + V_2\right) B_2^{-1} = i\hbar \frac{\partial B_2^{-1}}{\partial t} \tag{28}$$

With the definition

$$V_2 = \frac{-i\hbar}{m}\left(B_2 \nabla B_2^{-1} \cdot \nabla S_1 + \frac{\nabla^2 S_1}{2}\right) \tag{29}$$

Repeating the same procedure with B_2, one obtains

$$B_2^{-1} = e^{(i/\hbar)S_2}/B_3 \tag{30}$$

and equation (20) can thus be written as

$$B^{-1} \equiv B_1^{-1} = e^{(i/\hbar)(S_1+S_2)}/B_3 \tag{31}$$

Substituting this expression for B in equation (19), the kernel is given by

$$K = e^{(i/\hbar)(S_0+S_1+S_2)}/B_3 \tag{32}$$

Of course, this type of procedure can be carried out to any order to obtain

$$K = \frac{\exp[(i/\hbar)(S_0+S_1+\cdots+S_{N-1})]}{B_N} \tag{33}$$

Where each S_k satisfies the Hamilton–Jacobi equation:

$$\frac{(\nabla S_k)^2}{2m} + V_k = \frac{-\partial S_k}{\partial t} \tag{34}$$

with

$$V_k = \frac{-i\hbar}{m}\left(B_k \nabla B_k^{-1} \cdot \nabla S_{k-1} + \frac{\nabla^2 S_{k-1}}{2} \right) \tag{35}$$

B_N satisfies the Schrödinger equation

$$\left(\frac{-\hbar^2}{2m} \nabla^2 + V_N \right) B_N^{-1} = i\hbar \frac{\partial B_N^{-1}}{\partial t} \tag{36}$$

Each B_k can be written in terms of B_N $(k < N)$ as

$$B_k^{-1} = \frac{\exp[(i/\hbar)(S_k+S_{k+1}+\cdots+S_{N-1})]}{B_N} \tag{37}$$

At this point, it may seem that the problem is more complicated than it was originally in that the calculation of the kernel has been changed from the solution of one Schrödinger equation to the solution of N Hamilton–Jacobi equations in addition to one Schrödinger equation ($N = 1, 2, 3, \ldots$). An analysis of equations (34) and (35), however, shows that each successive V_k is proportional to a higher power of $\hbar$ (although the power may not be integral). This means that in terms of an $\hbar$ expansion, each successive V_k becomes smaller and smaller so that for some N, it is accurate to take

$V_N \simeq 0$. Then equation (36) become Schrödinger's equation for a free particle

$$\frac{-\hbar^2}{2m}\nabla^2 B_N^{-1} = i\hbar\frac{\partial B_N^{-1}}{\partial t} \tag{38}$$

The appropriate solution to this is easily shown to be

$$B_N^{-1} = e^{(i/\hbar)S_0^{\text{free}}}/B_{\text{free}} \tag{39}$$

Where S_0^{free} is the classical action for a free particle

$$S_0^{\text{free}} = \frac{m(\mathbf{x}-\mathbf{x}_0)^2}{2(t-t_0)} \tag{40}$$

and

$$B_{\text{free}} = \left[\frac{2\pi i\hbar(t-t_0)}{m}\right]^{3/2} \tag{41}$$

In fact, B_N^{-1} is just the kernel for a free particle

$$B_N^{-1} = K_{\text{free}} \tag{42}$$

so that equation (33) can be written as

$$K = K_{\text{free}}\exp[(i/\hbar)(S_0 + S_1 + \cdots + S_{N-1})] \tag{43}$$

Substituting equation (39) into equations (35) and (37), the "potentials" can be written as

$$V_k = \frac{-i\hbar}{m}K_{\text{free}}^{-1}\nabla K_{\text{free}}\cdot\nabla S_{k-1} + \frac{i}{\hbar}(\nabla S_k + \nabla S_{k+1} + \cdots + \nabla S_{N-1})$$
$$\times \nabla S_{k-1} + \frac{\nabla^2 S_{k-1}}{2} \tag{44}$$

Using these potentials in the Hamilton–Jacobi equation to obtain the various S_k's reduces the calculation of the kernel, finally, to the solution of a set of Hamilton–Jacobi equations. An iterative procedure is required, however, because to get V_k one needs S_k according to equation (44). But S_k is calculated using V_k.

Before justifying the iterative procedure to be followed, it is useful to examine the boundary conditions imposed upon the S_k's in the solutions to equation (34). In view of equation (43), the sum $S_0 + S_1 + \cdots + S_{N-1}$ must vanish in the limit of a free particle. Since N is arbitrary, it is evident that this means that each S_k must vanish in this limit:

$$S_k \to 0 \qquad (\text{free particle; } V_k \to 0) \tag{45}$$

where $k = 0, 1, 2, \ldots$. Other boundary conditions on the S_k's are to be determined from the boundary conditions on the kernel itself (Rosen, 1969). These conditions are: (1) the composition law

$$K(\mathbf{x}, t; \mathbf{x}_0, t_0) = \int K(\mathbf{x}, t; \mathbf{x}', t') K(\mathbf{x}', t'; \mathbf{x}_0, t_0)\, d\mathbf{x}' \tag{46}$$

for all $t_0 \leqslant t' \leqslant t$; and (2) the initial-value condition

$$\lim_{t \to t_0} K(\mathbf{x}, t; \mathbf{x}_0, t_0) = \delta^3(\mathbf{x} - \mathbf{x}_0) \tag{47}$$

Hence, assuming that the energy of the incident particle is large compared to the scattering potential so that the particle is almost free one can assume to first order that $S_1^{(1)} = S_2^{(1)} = \cdots = 0$ (which is equivalent to taking $N = 1$). This gives a first-order kernel as

$$K^{(1)} = K_{\text{free}} e^{(i/\hbar) S_0^{(1)}} \tag{48}$$

$S_0^{(1)}$ can be calculated because the physical scattering potential is used for its calculation. The reason for an iteration index on S_0 will become apparent in the following discussion.

The second-order approximation is obtained by taking $S_2^{(2)} = S_3^{(2)} = \cdots = 0$ (i.e., $N = 2$) and calculating V_1 using the first-order S_k's:

$$V_1^{(2)} = \frac{-i\hbar}{m} K_{\text{free}}^{-1} \nabla K_{\text{free}} \cdot \nabla S_0 + \frac{\nabla^2 S_0}{2} \tag{49}$$

This allows one to calculate $S_1^{(2)}$ by solving the Hamilton–Jacobi equation and to write K as

$$K^{(2)} = K_{\text{free}} e^{(i/\hbar)(S_0^{(2)} + S_1^{(2)})} \tag{50}$$

This procedure can then be continued indefinitely to obtain

$$K^{(n)} = K_{\text{free}} \exp\left[(i/\hbar)\left(S_0^{(n)} + S_1^{(n)} + S_2^{(n)} + \cdots + S_{N-1}^{(n)}\right)\right] \tag{51}$$

The above is an $\hbar$ expansion in that it corresponds to taking successively larger values for N, but it is also based on the assumption that the magnitude of the potential is much smaller than the energy of the incident particle. Hence, there are actually two expansion parameters that one must keep up with. These parameters can be taken to be the same as taken by Wallace (1973) in his eikonal expansion:

$$\varepsilon \equiv V_0/2E \tag{52}$$

and

$$\Delta \equiv \hbar/pa \tag{53}$$

Here V_0 is a measure of the magnitude of the strength of the potential, p is the magnitude of the particle momentum, and a is a measure of the distance over which the potential changes appreciably.

As discussed by Wallace, it is inconsistent to expand in only one of these parameters while ignoring the other. Hence, one must expand in both parameters together. For instance, terms like ε^2, Δ^2, and $\varepsilon\Delta$ should be kept in the same iteration step. It can now be understood why even S_0 needs an iteration index although it is not proportional to any power of $\hbar$. This is because it does contain terms with different orders of the parameter ε.

4. THE GLAUBER APPROXIMATION

The first-order solution to the kernel is given by

$$K^{(1)} = K_{\text{free}} e^{(i/\hbar)S_0^{(1)}} \tag{54}$$

Where S_0 satisfies the Hamilton–Jacobi equation

$$\frac{(\nabla S_0)^2}{2m} + V = \frac{-\partial S_0}{\partial t} \tag{55}$$

In the usual manner for time independent potentials, the time dependence of S_0 can be assumed to be of the form

$$S_0(\mathbf{x}, t; \mathbf{x}_0, t_0) = \bar{S}_0(\mathbf{x}, \mathbf{x}_0) - E(t - t_0) \tag{56}$$

so that

$$\frac{(\nabla \bar{S}_0)^2}{2m} + V = E \tag{57}$$

with E being the energy of the incident particle.

The solution to equation (57) normalized by equation (45) is known to be (Lanczos, 1966)

$$\bar{S}_0=\int_{-\infty}^{+\infty}\{[2m(E-V)]^{1/2}-(2mE)^{1/2}\}dz \tag{58}$$

Here the integral is done once the z direction from $-\infty$ to $+\infty$ because the Glauber approximation makes the assumption that the particle travels along a straight path; the direction on the path is defined to be the z direction and is the direction of the average momentum. The end points $\pm\infty$ correspond to the physical fact that the incident and scattered particles must be an infinite (out of range) distance from the potential center. This is also easily shown to be identical to the limits $t_0 \to -\infty$ and $t \to \infty$.

$\bar{S}_0$ can be expanded to give

$$\bar{S}_0=\sum_{j=1}^{\infty}\bar{S}_0^{(j)} \tag{59}$$

Where

$$\bar{S}_0^{(j)}=-(2mE)^{1/2}\frac{(2j-3)!!}{j!!}\int_{-\infty}^{\infty}\left(\frac{V}{2E}\right)^j dz \tag{60}$$

It can be seen from this that $\bar{S}_0^{(j)}$ is proportional to ε^j. Hence, the first-order solution to the kernel is

$$K^{(1)}=K_{\text{free}}e^{(-i/\hbar)E(t-t_0)}e^{(i/\hbar)\bar{S}_0^{(1)}} \tag{61}$$

Where

$$\bar{S}_0^{(1)}=-(2mE)^{1/2}\int_{-\infty}^{\infty}\frac{V}{2E}dz \tag{62}$$

If the "impact parameter" **b** is defined such that

$$\mathbf{x}=\mathbf{b}+z\hat{z} \tag{63}$$

then it is seen that

$$\bar{S}_0^{(j)}=\bar{S}_0^{(j)}(\mathbf{b}) \tag{64}$$

Substituting equation (61) into the expression for the S-matrix element, it is shown in the Appendix that the scattering amplitude is given by

$$f(\mathbf{p},\mathbf{p}_0)=\frac{p_0}{2\pi i\hbar}\int d^2b\; e^{i\mathbf{q}\cdot\mathbf{b}}(e^{i\chi(\mathbf{b})}-1) \tag{65}$$

where

$$\chi(\mathbf{b}) = \frac{\bar{S}_0^{(1)}}{\hbar} = \frac{-1}{\hbar}\left(\frac{m}{2E}\right)^{1/2}\int_{-\infty}^{\infty} V(\mathbf{b}, z)\, dz \tag{66}$$

and

$$\mathbf{q} = (\mathbf{p}_0 - \mathbf{p})/\hbar \tag{67}$$

This equation is identical to the Glauber approximation.

5. CONCLUSION

The calculation of the S-matrix element has been reduced to the solution of a set of classical Hamilton–Jacobi equations. The procedure outlined is expected to be convergent in the high-energy limit where eikonal approximations are useful. One advantage of such a formalism is that approximations based upon intuition gained from classical physics can be easily incorporated into the quantum theory.

In this regards, it was shown that the widely used Glauber approximation corresponds to the first term in the expansion. Its approximation of a straight-line trajectory was easily put into the formalism by calculating the classical action (the solution to the Hamilton–Jacobi equation) for such a path. Corrections to the Glauber approximation, therefore, can be obtained by retaining additional terms in the expansion. Such a procedure has been carried out and proved to be very promising. This will be the subject of a subsequent paper.

APPENDIX

From equations (2) and (3) the relationship between the S matrix and the kernel is

$$\langle \mathbf{p}|S|\mathbf{p}_0\rangle = e^{(i/\hbar)Et} e^{(-i/\hbar)E_0 t_0} \int \frac{d\mathbf{x}\, d\mathbf{x}_0}{|2\pi i\hbar|^2} e^{(-i/\hbar)\mathbf{p}\cdot\mathbf{x}} K(\mathbf{x}, t; \mathbf{x}_0, t_0) e^{(i/\hbar)p_0\cdot \mathbf{x}_0} \tag{A.1}$$

The kernel for the Glauber approximation is

$$K \simeq K_{\text{free}} e^{(-i/\hbar)E(t-t_0)} e^{i\chi(\mathbf{b}, E)} \tag{A.2}$$

where

$$\chi(\mathbf{b}, E) = \frac{-1}{\hbar}\left(\frac{m}{2E}\right)^{1/2}\int_{-\infty}^{+\infty} V(\mathbf{b}, z)\, dz \tag{A.3}$$

and

$$K_{\text{free}} = \exp\left[\frac{im(\mathbf{x}-\mathbf{x}_0)^2}{2\hbar(t-t_0)}\right] \Big/ \left[\frac{2\pi i\hbar(t-t_0)}{m}\right]^{3/2} \tag{A.4}$$

Setting $E = E_0$ in equation (A.1) and substituting for K, the S matrix becomes

$$\langle \mathbf{p}|S|\mathbf{p}_0\rangle = \int \frac{d\mathbf{x}\, d\mathbf{x}_0}{|2\pi i\hbar|^2} e^{(-i/\hbar)\mathbf{p}\cdot\mathbf{x}} K_{\text{free}} e^{i\chi} e^{(i/\hbar)\mathbf{p}_0\cdot\mathbf{x}_0} \tag{A.5}$$

In terms of the matrix element of $S-1$ this becomes

$$\langle \mathbf{p}|S-1|\mathbf{p}_0\rangle = \int \frac{d\mathbf{x}}{|2\pi i\hbar|^2} e^{i\mathbf{q}\cdot\mathbf{x}} T(\mathbf{b}, E) \tag{A.6}$$

Where

$$T(\mathbf{b}, E) = e^{(-i/\hbar)\mathbf{p}_0\cdot\mathbf{x}} \int d\mathbf{x}_0\, e^{(i/\hbar)\mathbf{p}_0\cdot\mathbf{x}_0} K_{\text{free}} e^{i\chi} - 1 \tag{A.7}$$

and $\mathbf{q}$ is the momentum transfer given by

$$\mathbf{q} = (\mathbf{p}_0 - \mathbf{p})/\hbar \tag{A.8}$$

Using equation (A.4) for K_{free}, it is a straightforward calculation to show that

$$T(\mathbf{b}, E) = e^{i\chi(\mathbf{b}, E)} - 1 \tag{A.9}$$

Substituting equation (A.6) into equation (12) of Section 2, the cross section becomes

$$\sigma = d\Omega \int \frac{p^2 dp\, d^2\rho\, d\mathbf{p}'\, d\mathbf{p}''\, d\mathbf{x}\, d\mathbf{x}''}{|2\pi i\hbar|^4} e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}/\hbar} e^{-i(\mathbf{p}''-\mathbf{p})\cdot\mathbf{x}''/\hbar}$$

$$\times\, e^{i(\mathbf{p}''-\mathbf{p}')\cdot\rho} \phi_{\mathbf{p}_0}(\mathbf{p}') \phi^*_{\mathbf{p}_0}(\mathbf{p}'') T(E, \mathbf{b}) T^*(E, \mathbf{b}) \tag{A.10}$$

The integral over $\boldsymbol{\rho}$ can be done to yield a two-dimensional delta function in momentum space corresponding to the momentum components in the plane perpendicular to $\mathbf{p}_0$:

$$\int d^2\rho\, e^{i(\mathbf{p}''-\mathbf{p}')\cdot\boldsymbol{\rho}/\hbar} = |2\pi i\hbar|^2\delta^2(\mathbf{p}''_+ - \mathbf{p}'_-) \qquad (A.11)$$

Where

$$\mathbf{p}' = \mathbf{p}'_+ + p'_-\,\hat{p}_0 \qquad (A.12)$$

and

$$\mathbf{p}'' = \mathbf{p}''_+ + p'_-\,\hat{p}_0 \qquad (A.13)$$

define $\mathbf{p}'_+$, $\mathbf{p}''_+$, p'_-, and p''_-. This delta function can then be used to do the integral over $d^2p''_+$ to obtain

$$\sigma = d\Omega\int\frac{p^2dp\,d\mathbf{p}'\,dp''_-\,d\mathbf{x}\,d\mathbf{x}}{|2\pi i\hbar|^4}\exp[i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}/\hbar]\exp[i(\mathbf{p}'_+ + p''_-\,\hat{p}_0-\mathbf{p})\cdot\mathbf{x}'']$$
$$\times\,\phi_{\mathbf{p}_0}(\mathbf{p}')\phi^*_{\mathbf{p}_0}(\mathbf{p}'_+ + p''_-\,\hat{p}_0)T(E,\mathbf{b})T^*(E,\mathbf{b}) \qquad (A.14)$$

The integrals over z and z'' can also be done obtaining two delta functions:

$$\sigma = d\Omega\int\frac{p^2\,d\mathbf{p}\,d\mathbf{p}'\,d\mathbf{p}''\,d^2b\,d^2b''}{|2\pi i\hbar|^2}\delta((\mathbf{p}'-\mathbf{p})\cdot\hat{z})\delta((\mathbf{p}'_+ + p''_-\,\hat{p}_0-\mathbf{p})\cdot\hat{z})$$
$$\times\exp[i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{b}/\hbar]\exp[-i(\mathbf{p}'_+ + p''_-\,\hat{p}_0-\mathbf{p})\cdot\mathbf{b}''/\hbar]$$
$$\times\,\phi_{\mathbf{p}_0}(\mathbf{p}')\phi^*_{\mathbf{p}_0}(\mathbf{p}'_+ + p''_-\,\hat{p}_0)T(E,\mathbf{b})T^*(E,\mathbf{b}'') \qquad (A.15)$$

Using the fact that $\delta(ax)=\delta(x)/|a|$, one can write

$$\delta((\mathbf{p}'_+ + p''_-\,\hat{p}_0-\mathbf{p})\cdot\hat{z}) = (p_0/|g|)\delta(p''_- -(\mathbf{p}-\mathbf{p}'_+)\cdot\hat{z}p_0/g) \qquad (A.16)$$

where

$$g \equiv \mathbf{p}_0\cdot z \qquad (A.17)$$

This delta function can be used to do the integral over p''_- to obtain

$$\sigma = d\Omega \int \frac{p^2 dp\, d^2b\, d^2b''\, d\mathbf{p}'}{|2\pi i\hbar|^2} \frac{p_0}{|g|} \delta((\mathbf{p}'-\mathbf{p})\cdot\hat{z}) \exp[i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{b}/\hbar]$$
$$\times \exp\{-i[\mathbf{p}'_+ + (\mathbf{p}-\mathbf{p}'_+)\cdot\hat{z}\mathbf{p}_0/g - \mathbf{p}]\cdot\mathbf{b}''/\hbar\}$$
$$\times \phi_{\mathbf{p}_0}(\mathbf{p}')\phi^*_{\mathbf{p}_0}[\mathbf{p}'_+ + (\mathbf{p}-\mathbf{p}'_+)\cdot\hat{z}\hat{p}_0/g] T(E,\mathbf{b})T^*(E,\mathbf{b}'') \tag{A.18}$$

The delta function appearing in this equation causes

$$\mathbf{p}'_+ + (\mathbf{p}-\mathbf{p}'_+)\cdot\hat{z}\mathbf{p}_0/\mathbf{p}_0\cdot\hat{z} = \mathbf{p}'_+ + p'_-\hat{p}_0 = \mathbf{p} \tag{A.19}$$

so that the $\mathbf{p}'$ integration is of the form

$$\int d\mathbf{p}'\delta((\mathbf{p}'-\mathbf{p})\cdot\hat{z})|\phi_{\mathbf{p}_0}(\mathbf{p}')|^2 \exp[i(\mathbf{p}'-\mathbf{p})\cdot(\mathbf{b}-\mathbf{b}'')/\hbar]$$
$$= \exp[i(\mathbf{p}_0-\mathbf{p})\cdot(\mathbf{b}-\mathbf{b}'')/\hbar] \int d\mathbf{p}'\delta((\mathbf{p}'-\mathbf{p})\cdot\hat{z})|\phi_{\mathbf{p}_0}(\mathbf{p}')|^2$$
$$\times \exp[i(\mathbf{p}'-\mathbf{p}_0)\cdot(\mathbf{b}-\mathbf{b}'')/\hbar] \tag{A.20}$$

In the plane wave limit of $|\phi|^2 \to \delta^3(\mathbf{p}'-\mathbf{p}_0)$, the integral above reduces to $\delta((\mathbf{p}_0-\mathbf{p})\cdot\hat{z})\exp[i(\mathbf{p}_0-\mathbf{p})\cdot(\mathbf{b}-\mathbf{b}'')/\hbar]$. Substituting this result into equation (A.18), one obtains for the cross section

$$\sigma = d\Omega \int \frac{p^2 dp\, d^2b\, d^2b''}{|2\pi i\hbar|^2} \frac{p_0}{|g|} \delta((\mathbf{p}_0-\mathbf{p})\cdot\hat{z})$$
$$\times \exp[i(\mathbf{p}_0-\mathbf{p})\cdot(\mathbf{b}-\mathbf{b}'')/\hbar] T(E,\mathbf{b})T^*(E,\mathbf{b}'') \tag{A.21}$$

The $\hat{z}$ direction for the Glauber approximation is the direction of the average momentum:

$$\hat{z} = \frac{\mathbf{p}+\mathbf{p}_0}{|\mathbf{p}+\mathbf{p}_0|} = \frac{\mathbf{p}+\mathbf{p}_0}{2p_0\cos\theta/2} \tag{A.22}$$

Where θ is the scattering angle. Hence,

$$(\mathbf{p}_0-\mathbf{p})\cdot\hat{z} = \frac{m}{p_0\cos\theta/2}\frac{p_0^2-p^2}{2m}$$
$$= \frac{m}{p_0\cos\theta/2}(E_0-E) \tag{A.23}$$

But

$$g = \mathbf{p}_0 \cdot \hat{z} = \mathbf{p}_0 \cdot \frac{(\mathbf{p}_0 + \mathbf{p})}{2 p_0 \cos\theta/2} = p_0 \cos\theta/2 \tag{A.24}$$

This gives

$$(\mathbf{p}_0 - \mathbf{p}) \cdot \hat{z} = \frac{m}{g}(E_0 - E) \tag{A.25}$$

and

$$\delta((\mathbf{p}_0 - \mathbf{p}) \cdot \hat{z}) = \delta\left(\frac{m}{g}(E_0 - E)\right) = \frac{|g|}{m}\delta(E_0 - E) \tag{A.26}$$

The cross section becomes

$$\sigma = d\Omega \int \frac{p_0 p^2 dp}{|2\pi i \hbar|^2} \frac{\delta(E_0 - E)}{m} \exp\left[i(\mathbf{p}_0 - \mathbf{p}) \cdot (\mathbf{b} - \mathbf{b}'')/\hbar\right] T(E, \mathbf{b}) T^*(E, \mathbf{b}'') \tag{A.27}$$

Using $dp = (m/p)dE$ and the delta function to do the integral, equation (A.27) becomes

$$\sigma = d\Omega \left| \frac{p_0}{2\pi i \hbar} \int d^2 b\, e^{i\mathbf{q} \cdot \mathbf{b}} T(E, \mathbf{b}) \right|^2$$

Upon comparing this with the equation

$$\sigma = d\Omega |f(\mathbf{p}, \mathbf{p}_0)|^2$$

it is seen that the scattering amplitude is given by

$$f(\mathbf{p}, \mathbf{p}_0) = \frac{p_0}{2\pi i \hbar} \int d^2 b\, e^{i\mathbf{q} \cdot \mathbf{b}} T(E, \mathbf{b})$$

$$= \frac{p_0}{2\pi i \hbar} \int d^2 b\, e^{i\mathbf{q} \cdot \mathbf{b}} (e^{i\chi(\mathbf{b})} - 1)$$

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